

Statistical immersions between statistical manifolds of constant curvature

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Abstract

The condition for the curvature of a statistical manifold to admit a kind of standard hypersurface is given. We study the statistical hypersurfaces of some types of the statistical manifolds (M, ∇, g) , which enable $(M, \nabla^{(\alpha)}, g), \forall \alpha \in \mathbf{R}$ to admit the structure of a constant curvature.

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1 Introduction

Since Lauritzen introduced the notation of statistical manifolds in 1987 [5], the geometry of statistical manifolds has been developed in close relations with affine differential geometry and Hessian geometry as well as information geometry (see, for example, [2, 4, 8]). In this paper we study the hypersurface of statistical manifolds.

Let M be an n -dimensional manifold, ∇ a torsion-free affine connection on M , g a Riemannian metric on M , and R a curvature tensor field on M . We denote by TM the set of vector fields on M , and by $TM^{(r,s)}$ the set of tensor fields of type (r, s) on M .

Definition 1.1. A pair (∇, g) is called a *statistical structure* on M if (M, ∇, g) is a statistical manifold, that is, ∇ is a torsion-free affine connection and for all $X, Y, Z \in T(M)$, $(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$.

Let ∇° be a Levi-Civita connection of g . Certainly, a pair (∇°, g) is a statistical structure, which is called a Riemannian statistical structure or a trivial statistical structure (see [3]).

On the other hand, the statistical structure is also introduced from affine differential geometry which was proposed by Blasche (see [6]). Recently the relation between statistical structures and Hessian geometry has been studied (see [3, 7]).

For all $\alpha \in \mathbf{R}$, a connection $\nabla^{(\alpha)}$ is defined by

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*$$

where ∇ and ∇^* are dual connections on M . We study a statistical hypersurface of a statistical manifold (M, ∇, g) which enables $(M, \nabla^{(\alpha)}, g), \forall \alpha \in \mathbf{R}$ to admit the structure of a constant curvature.

In section 3, a statistical manifold (M, ∇, g) , which enables $(M, \nabla^{(\alpha)}, g), \forall \alpha \in \mathbf{R}$ to admit the structure of a constant curvature, is considered. In section 4, we study characteristics of statistical immersions between statistical manifolds (M, ∇, g) which enable $(M, \nabla^{(\alpha)}, g), \forall \alpha \in \mathbf{R}$ to admit the structure of a constant curvature.

2 Preliminaries

A statistical manifold (M, ∇, g) is said to be of constant curvature $k \in \mathbf{R}$ if

$$R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\}, \forall X, Y, Z \in TM \quad (2.1)$$

holds, where R is the curvature tensor field of ∇ . A pair (∇, g) is called a Hessian structure if a statistical manifold (M, ∇, g) is of constant curvature 0.

A Riemannian metric g on a flat manifold (M, g) is called a Hessian metric if g can be locally expressed by

$$g = Dd\varphi,$$

that is,

$$g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j},$$

where $\{x^1, \dots, x^n\}$ is an affine coordinate system with respect to ∇ . Then (M, ∇, g) is called a Hessian manifold (see [7]).

Let (M, ∇, g) be a Hessian manifold and $K(X, Y) := \nabla_X Y - \nabla_X^\circ Y$ be the difference tensor between the Levi-Civita connection ∇° of g and ∇ . A covariant differential of differential tensor K is called a Hessian curvature tensor for (∇, g) . A Hessian manifold (M, ∇, g) is said to be of constant Hessian curvature $c \in \mathbf{R}$ if

$$(\nabla_X K)(Y, Z) = -\frac{c}{2}\{g(X, Y)Z + g(X, Z)Y\}, \forall X, Y, Z \in TM$$

holds (see [7]).

Example 2.1.([3]) Let (H, \tilde{g}) be the upper half space:

$$H := \{y = (y^1, \dots, y^{n+1})^T \in \mathbf{R}^{n+1} \mid y^{n+1} > 0\}, \tilde{g} := (y^{n+1})^{-2} \sum_{i=1}^{n+1} dy^i dy^i.$$

We define an affine connection $\tilde{\nabla}$ on H by the following relations:

$$\begin{aligned} \tilde{\nabla}_{\frac{\partial}{\partial y^{n+1}}} \frac{\partial}{\partial y^{n+1}} &= (y^{n+1})^{-1} \frac{\partial}{\partial y^{n+1}}, \quad \tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 2\delta_{ij}(y^{n+1})^{-1} \frac{\partial}{\partial y^{n+1}}, \\ \tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^{n+1}} &= \tilde{\nabla}_{\frac{\partial}{\partial y^{n+1}}} \frac{\partial}{\partial y^j} = 0, \end{aligned}$$

where $i, j = 1, \dots, n$. Then $(H, \tilde{\nabla}, \tilde{g})$ is a Hessian manifold of constant Hessian curvature 4.

Let $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ be a statistical manifold and $f : M \rightarrow \tilde{M}$ be an immersion. We define g and ∇ on M by

$$g = f^* \tilde{g}, \quad g(\nabla_X Y, Z) = \tilde{g}(\tilde{\nabla}_X f_* Y, f_* Z), \quad \forall X, Y, Z \in TM.$$

Then the pair (∇, g) is a statistical structure on M , which is called the one by f from $(\tilde{\nabla}, \tilde{g})$ (see [3]).

Let (M, ∇, g) and $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ be two statistical manifolds. An immersion $f : M \rightarrow \tilde{M}$ is called a statistical immersion if $(\tilde{\nabla}, \tilde{g})$ coincides with the induced statistical structure (see [3]).

Let $f : (M, \nabla, g) \rightarrow (\tilde{M}, \tilde{\nabla}, \tilde{g})$ be a statistical immersion of codimension one, and ξ a unit normal vector field of f . Then we define $h, h^* \in TM^{(0,2)}$, $\tau, \tau^* \in TM^*$ and $A, A^* \in TM^{(1,1)}$ by the following Gauss and Weingarten formulae:

$$\tilde{\nabla}_X f_* Y = f_* \nabla_X Y + h(X, Y)\xi, \quad \tilde{\nabla}_X \xi = -f_* A^* X + \tau^*(X)\xi,$$

$$\tilde{\nabla}_X^* f_* Y = f_* \nabla_X^* Y + h^*(X, Y)\xi, \quad \tilde{\nabla}_X^* \xi = -f_* A X + \tau(X)\xi, \quad \forall X, Y \in TM,$$

where $\tilde{\nabla}^*$ is the dual connection of $\tilde{\nabla}$ with respect to \tilde{g} .

In addition, we define $II \in TM^{(0,2)}$ and $S \in TM^{(1,1)}$ by using the Riemannian Gauss and Weingarten formulae:

$$\tilde{\nabla}_X^* f_* Y = f_* \nabla_X^* Y + II(X, Y)\xi, \quad \tilde{\nabla}_X^* \xi = -f_* S X.$$

For more details on the Gauss, Codazzi and Ricci formulae on statistical hypersurfaces, we refer to [3].

3 The condition that a statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature for any $\alpha \in \mathbf{R}$

In this section we consider a condition that a statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature for any $\alpha \in \mathbf{R}$.

Theorem 3.1. *A statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature for any $\alpha \in \mathbf{R}$ iff there exist $\alpha_1, \alpha_2 \in \mathbf{R} (|\alpha_1| \neq |\alpha_2|)$ such that statistical manifolds $(M, \nabla^{(\alpha_1)}, g)$ and $(M, \nabla^{(\alpha_2)}, g)$ are of constant curvature.*

Proof. Necessity is obvious. We find sufficiency. Without loss of generality, we assume $\alpha_1 \neq 0$. Then since

$$\nabla^{(\alpha)} = \frac{\alpha_1 + \alpha}{2\alpha_1} \nabla^{(\alpha_1)} + \frac{\alpha_1 - \alpha}{2\alpha_1} \nabla^{(-\alpha_1)}$$

holds for all $\alpha \in \mathbf{R}$, the following relation

$$\begin{aligned} R^{(\alpha)}(X, Y)Z &= \frac{\alpha_1 + \alpha}{2\alpha_1} R^{(\alpha_1)}(X, Y)Z + \frac{\alpha_1 - \alpha}{2\alpha_1} R^{(-\alpha_1)}(X, Y)Z \\ &\quad + \frac{\alpha_1^2 - \alpha^2}{4\alpha_1^2} [K(Y, K(Z, X)) - K(X, K(Y, Z))] \end{aligned}$$

holds, where $K(X, Y) := \nabla_X Y - \nabla_Y X$ is the difference tensor field of a statistical manifold.

From the relations

$$R^{(\alpha_1)}(X, Y)Z = k_1 \{g(Y, Z)X - g(X, Z)Y\},$$

$$R^{(\alpha_2)}(X, Y)Z = k_2 \{g(Y, Z)X - g(X, Z)Y\},$$

the relation

$$R^{(\alpha)}(X, Y)Z = \frac{k_2 \alpha_1^2 - k_1 \alpha_2^2 + (k_1 - k_2) \alpha^2}{\alpha_1^2 - \alpha_2^2} \{g(Y, Z)X - g(X, Z)Y\}$$

holds, that is, a statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature $\frac{k_2 \alpha_1^2 - k_1 \alpha_2^2 + (k_1 - k_2) \alpha^2}{\alpha_1^2 - \alpha_2^2}$. \square

Example 3.1. Let (M, g) be a family of normal distributions:

$$M := \left\{ p(x, \theta) \left| p(x, \theta) = \frac{1}{\sqrt{2\pi(\theta^2)^2}} \exp \left\{ -\frac{1}{2(\theta^2)^2} (x - \theta^1)^2 \right\} \right. \right\}, \quad g := 2(\theta^2)^{-2} \sum d\theta^i d\theta^i,$$

$$x \in \mathbf{R}, \quad \theta^1 \in \mathbf{R}, \quad \theta^2 > 0.$$

We define an α -connection by the following relations:

$$\nabla_{\frac{\partial}{\partial \theta^1}}^{(\alpha)} \frac{\partial}{\partial \theta^1} = (-1 + 2\alpha)(\theta^2)^{-1} \frac{\partial}{\partial \theta^2}, \quad \nabla_{\frac{\partial}{\partial \theta^2}}^{(\alpha)} \frac{\partial}{\partial \theta^2} = (1 + \alpha)(\theta^2)^{-1} \frac{\partial}{\partial \theta^2},$$

$$\nabla_{\frac{\partial}{\partial \theta^1}}^{(\alpha)} \frac{\partial}{\partial \theta^2} = \nabla_{\frac{\partial}{\partial \theta^2}}^{(\alpha)} \frac{\partial}{\partial \theta^1} = 0.$$

Then the statistical manifold $(M, \nabla^{(0)}, g)$ is of constant curvature $(-\frac{1}{2})$, and the statistical manifold $(M, \nabla^{(1)}, g)$ is of constant curvature 0. Hence for all $\alpha \in \mathbf{R}$, the statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature $\frac{\alpha^2 - 1}{2}$.

Example 3.2. Let (M, g) be a family of random walk distributions ([1]):

$$M := \left\{ p(x; \theta^1, \theta^2) \left| p(x; \theta^1, \theta^2) = \sqrt{\frac{\theta^2}{2\pi x}} \exp \left\{ -\frac{\theta^2 x}{2} + \frac{\theta^2}{\theta^1} - \frac{\theta^2}{2(\theta^1)^2 x} \right\}, \quad x, \mu, \lambda > 0 \right. \right\},$$

$$g := \frac{\theta^2}{(\theta^1)^3} (d\theta^1)^2 + \frac{1}{2(\theta^2)^2} (d\theta^2)^2.$$

We define an α -connection by the following relations:

$$\nabla_{\frac{\partial}{\partial \theta^1}}^{(\alpha)} \frac{\partial}{\partial \theta^1} = \frac{-3(1 + \alpha)}{2} (\theta^1)^{-1} \frac{\partial}{\partial \theta^1} + (-1 + \alpha)(\theta^1)^{-3} (\theta^2)^2 \frac{\partial}{\partial \theta^2},$$

$$\nabla_{\frac{\partial}{\partial \theta^1}}^{(\alpha)} \frac{\partial}{\partial \theta^2} = \nabla_{\frac{\partial}{\partial \theta^2}}^{(\alpha)} \frac{\partial}{\partial \theta^1} = \frac{(1 + \alpha)}{2} (\theta^2)^{-1} \frac{\partial}{\partial \theta^1},$$

$$\nabla_{\frac{\partial}{\partial \theta^2}}^{(\alpha)} \frac{\partial}{\partial \theta^2} = (-1 + \alpha)(\theta^2)^{-1} \frac{\partial}{\partial \theta^2}.$$

Then the statistical manifold $(M, \nabla^{(0)}, g)$ is of constant curvature $(-\frac{1}{2})$, and the statistical manifold $(M, \nabla^{(1)}, g)$ is of constant curvature 0. Hence for all $\alpha \in \mathbf{R}$, the statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature $\frac{\alpha^2 - 1}{2}$.

Theorem 3.1 implies the following fact.

Corollary 3.1. *If there exist $\alpha_1, \alpha_2 \in \mathbf{R}$ ($|\alpha_1| \neq |\alpha_2|$) such that the statistical manifold $(M, \nabla^{(\alpha_1)}, g)$ is of constant curvature k_1 and the statistical manifold $(M, \nabla^{(\alpha_2)}, g)$ is of constant curvature k_2 , and $k_1 \neq k_2$, then for $\alpha \in \mathbf{R}$ satisfying that $\alpha^2 = (k_2 \alpha_1^2 - k_1 \alpha_2^2) / (k_2 - k_1)$, the statistical manifold $(M, \nabla^{(\alpha)}, g)$ is flat.*

Example 3.3. $k_1 = -1/2$, $k_2 = 0$, $\alpha_1 = 0$ and $\alpha_2 = 1$ hold in example 3.1 and example 3.2. Hence for $\alpha \in \mathbf{R}$ satisfying that $\alpha^2 = 1$, the statistical manifold $(M, \nabla^{(\alpha)}, g)$ is flat.

Theorem 3.2. *If the Hessian manifold (M, ∇, g) is of constant Hessian curvature, then for all $\alpha \in \mathbf{R}$, the statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature.*

Proof. If the Hessian manifold (M, ∇, g) is of constant Hessian curvature, then for all $X, Y, Z \in TM$,

$$(\nabla K)(Y, Z; X) = -\frac{c}{2}\{g(X, Y)Z + g(X, Z)Y\}, c \in \mathbf{R}$$

holds. On the other hand, the curvature tensor R° of Levi-Civita connection ∇° is written by

$$\begin{aligned} R^\circ(X, Y)Z &= R(X, Y)Z - (\nabla K)(Y, Z; X) + (\nabla K)(Z, X; Y) \\ &\quad + K(X, K(Y, Z)) - K(Y, K(Z, X)), \end{aligned}$$

where R is the curvature tensor of ∇ and $K(X, Y) = \nabla_X Y - \nabla_X^\circ Y$ is difference tensor. Then

$$\begin{aligned} &(\nabla K)(Y, Z; X) - (\nabla K)(Z, X; Y) \\ &= 2\{K(X, K(Y, Z)) - K(Y, K(Z, X))\} + \frac{1}{2}\{R(X, Y)Z - R^*(X, Y)Z\} \end{aligned}$$

implies

$$R^\circ(X, Y)Z = -\frac{c}{4}\{g(Y, Z)X - g(X, Z)Y\},$$

where R^* is curvature tensor of dual connection ∇^* , that is, the statistical manifold (M, ∇°, g) is of constant curvature. On the other hand, the statistical manifold (M, ∇, g) is flat, that is, constant curvature 0. Therefore we finish the proof of theorem by applying Theorem 3.1. \square

Hitherto we found some conditions that for any $\alpha \in \mathbf{R}$, the statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature.

4 The hypersurfaces of statistical manifolds of constant curvature

We consider statistical hypersurfaces of some type of statistical manifolds, which enable for any $\alpha \in \mathbf{R}$ a statistical manifold $(M, \nabla^{(\alpha)}, g)$ to be of constant curvature.

Theorem 4.1. *Let (M, ∇, g) be a trivial statistical manifold of constant curvature k , $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ a statistical manifold of constant curvature \tilde{k} with a Riemannian manifold of constant curvature $\tilde{\tilde{k}} (\neq \tilde{k})$ $(\tilde{M}, \tilde{\nabla}^\circ, \tilde{g})$, and $f : M \rightarrow \tilde{M}$ a statistical immersion of codimension one. Then $f : M \rightarrow \tilde{M}$ is equiaffine, that is, τ^* vanishes.*

Proof. If $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ is a statistical manifold of constant curvature \tilde{k} with a Riemannian manifold of constant curvature $\tilde{\tilde{k}} (\neq \tilde{k})$ $(\tilde{M}, \tilde{\nabla}^\circ, \tilde{g})$, the following equation

$$\begin{aligned} &(\tilde{\nabla}_X \tilde{K})(f_* Y, f_* Z) - (\tilde{\nabla}_Y \tilde{K})(f_* X, f_* Z) \\ &= 2\{\tilde{R}(f_* X, f_* Y)f_* Z - \tilde{R}^\circ(f_* X, f_* Y)f_* Z\} \\ &= 2(\tilde{\tilde{k}} - \tilde{k})\{\tilde{g}(f_* Y, f_* Z)f_* X - \tilde{g}(f_* X, f_* Z)f_* Y\} \end{aligned} \tag{4.1}$$

holds by Eq.(2.2) and Eq.(2.3) in [3]. By above equation and equation Eq.(3.6) in [3], we have

$$\begin{aligned}
& -2(\tilde{k} - \overset{\circ}{\tilde{k}})\{g(Y, Z)X - g(X, Z)Y\} = (\nabla_X K)(Y, Z) - (\nabla_Y K)(X, Z) \\
& \quad -b(Y, Z)A^*X + b(X, Z)A^*Y + h(X, Z)B^*Y - h(Y, Z)B^*X \\
0 & = (\nabla_X b)(Y, Z) - (\nabla_Y b)(X, Z) + \tau^*(X)b(Y, Z) - \tau^*(Y)b(X, Z) \\
& \quad -\tau^*(Y)h(X, Z) + \tau^*(X)h(Y, Z) \\
0 & = -\tau^*(Y)A^*X + \tau^*(X)A^*Y - (\nabla_X B^*)Y + (\nabla_Y B^*)X + \tau^*(X)B^*Y - \tau^*(Y)B^*X \\
0 & = -h(X, B^*Y) + h(Y, B^*X) + (\nabla_X \tau^*)(Y) - (\nabla_Y \tau^*)(X) + b(Y, A^*X) - b(X, A^*Y).
\end{aligned} \tag{4.2}$$

By $K = 0$, $B^* = A^* - S$ and Gauss equation (3.3)₁ in [3], from Eq.(4.2)₁, we have

$$\begin{aligned}
& -2(\tilde{k} - \overset{\circ}{\tilde{k}})\{g(Y, Z)X - g(X, Z)Y\} = -b(Y, Z)A^*X + b(X, Z)A^*Y \\
& \quad + h(X, Z)A^*Y - h(X, Z)SY - h(Y, Z)A^*X + h(Y, Z)SX \\
& = -b(Y, Z)A^*X + b(X, Z)A^*Y - h(X, Z)SY + h(Y, Z)SX + \tilde{R}(X, Y)Z - R(X, Y)Z.
\end{aligned}$$

By $b = h - II$, $B^* = A^* - S$ and Riemannian Gauss equation (3.5)₁ in [3], we have

$$\begin{aligned}
& -2(\tilde{k} - \overset{\circ}{\tilde{k}})\{g(Y, Z)X - g(X, Z)Y\} \\
& = -h(Y, Z)A^*X + II(Y, Z)A^*X + h(X, Z)A^*Y - II(X, Z)A^*Y \\
& \quad - h(X, Z)SY + h(Y, Z)SX + \tilde{R}(X, Y)Z - R(X, Y)Z \\
& = -h(Y, Z)B^*X + h(X, Z)B^*Y + II(Y, Z)B^*X + II(Y, Z)SX \\
& \quad - II(X, Z)B^*Y - II(X, Z)SY + \tilde{R}(X, Y)Z - R(X, Y)Z \\
& = -b(Y, Z)B^*X + b(X, Z)B^*Y + R^\circ(X, Y)Z - \tilde{R}^\circ(X, Y)Z + \tilde{R}(X, Y)Z - R(X, Y)Z.
\end{aligned}$$

Since (M, ∇, g) is Riemannian manifold, clearly $R^\circ(X, Y)Z = R(X, Y)Z$. Hence we have

$$0 = (\tilde{k} - \overset{\circ}{\tilde{k}})\{g(Y, Z)X - g(X, Z)Y\} - b(Y, Z)B^*X + b(X, Z)B^*Y.$$

And since $b(Y, Z) = g(BY, Z)$, $b(X, Z) = g(BX, Z)$, from above equation we have

$$0 = (\tilde{k} - \overset{\circ}{\tilde{k}})\{g(Y, Z)X - g(X, Z)Y\} - g(BY, Z)B^*X + g(BX, Z)B^*Y. \tag{4.3}$$

From Eq.(4.2)₃, $B^* = A^* - S$ and Codazzi equation on A we get

$$\begin{aligned}
0 & = -\tau^*(Y)A^*X + \tau^*(X)A^*Y - (\nabla_X A^*)Y + (\nabla_X S)Y + (\nabla_Y A^*)X - (\nabla_Y S)X \\
& \quad + \tau^*(X)B^*Y - \tau^*(Y)B^*X \\
& = (\nabla_X S)Y - (\nabla_Y S)X + \tau^*(X)B^*Y - \tau^*(Y)B^*X
\end{aligned}$$

and by $\nabla = \nabla^\circ$ and Codazzi equation on S , we also get

$$0 = \tau^*(X)B^*Y - \tau^*(Y)B^*X. \tag{4.4}$$

From Eq.(4.2)₄, $B^* = A^* - S$ and Ricci equation we have

$$b(X, B^*Y) - b(Y, B^*X) = 0,$$

and since $b(X, B^*Y) = g(BX, B^*Y)$ and $b(Y, B^*X) = g(BY, B^*X)$, we have

$$g(BX, B^*Y) - g(BY, B^*X) = 0.$$

Since $g(BX, B^*Y) = g(B^*Y, BX) = b^*(BX, Y) = g(B^*BX, Y)$, we have

$$0 = -g([B, B^*]X, Y). \quad (4.5)$$

From Eq.(4.5), B and B^* are simultaneously diagonalizable.

In the case that B^* is of the form λ^*I , we see easily that τ^* vanishes from Eq.(4.4) if $\lambda^* \neq 0$ and $\tilde{k} = \overset{\circ}{k}$ from Eq.(4.3) otherwise. In the case that B^* is not of the form λ^*I , there are λ_1^*, λ_2^* with $\lambda_1^* \neq \lambda_2^*$ such that $B^*X_j = \lambda_j^*X_j$, where $g(X_i, X_j) = \delta_{ij}$, $i, j = 1, 2$. Besides there are λ_1, λ_2 such that $BX_j = \lambda_jX_j$. Eq.(4.3) implies that

$$\begin{aligned} & (\tilde{k} - \overset{\circ}{k})\{g(X_j, Z)X_i - g(X_i, Z)X_j\} + \lambda_j\lambda_i^*g(X_j, Z)X_i - \lambda_i\lambda_j^*g(X_i, Z)X_j \\ & = (\tilde{k} - \overset{\circ}{k} + \lambda_j\lambda_i^*)g(X_j, Z)X_i - (\tilde{k} - \overset{\circ}{k} + \lambda_i\lambda_j^*)g(X_i, Z)X_j = 0 \end{aligned}$$

and hence $\tilde{k} - \overset{\circ}{k} + \lambda_j\lambda_i^* = \tilde{k} - \overset{\circ}{k} + \lambda_i\lambda_j^* = 0$, which means that

$$\lambda_j\lambda_i^* = \lambda_i\lambda_j^* = -(\tilde{k} - \overset{\circ}{k}) \neq 0.$$

By Eq.(4.4) we have $\lambda_2^*\tau^*(X_1)X_2 - \lambda_1^*\tau^*(X_2)X_1 = 0$, which implies that τ^* vanishes. \square

Example 4.1. Suppose \tilde{M} be \mathbf{R}^3 . We define Riemannian metric and an Affine connection by the following relations:

$$\begin{aligned} \tilde{g} &= a \sum d\theta^i d\theta^i, \\ \tilde{\nabla}_{\frac{\partial}{\partial\theta^1}} \frac{\partial}{\partial\theta^1} &= \tilde{b} \frac{\partial}{\partial\theta^1}, \tilde{\nabla}_{\frac{\partial}{\partial\theta^2}} \frac{\partial}{\partial\theta^2} = \frac{\tilde{b}}{2} \frac{\partial}{\partial\theta^1}, \tilde{\nabla}_{\frac{\partial}{\partial\theta^3}} \frac{\partial}{\partial\theta^3} = \frac{\tilde{b}}{2} \frac{\partial}{\partial\theta^1}, \\ \tilde{\nabla}_{\frac{\partial}{\partial\theta^1}} \frac{\partial}{\partial\theta^2} &= \tilde{\nabla}_{\frac{\partial}{\partial\theta^2}} \frac{\partial}{\partial\theta^1} = \frac{\tilde{b}}{2} \frac{\partial}{\partial\theta^2}, \tilde{\nabla}_{\frac{\partial}{\partial\theta^1}} \frac{\partial}{\partial\theta^3} = \tilde{\nabla}_{\frac{\partial}{\partial\theta^3}} \frac{\partial}{\partial\theta^1} = \frac{\tilde{b}}{2} \frac{\partial}{\partial\theta^2}, \\ \tilde{\nabla}_{\frac{\partial}{\partial\theta^2}} \frac{\partial}{\partial\theta^3} &= \tilde{\nabla}_{\frac{\partial}{\partial\theta^3}} \frac{\partial}{\partial\theta^2} = 0. \end{aligned}$$

Then $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ is a statistical manifold of constant curvature $-\frac{\tilde{b}^2}{4a}$ with a trivial statistical manifold of constant curvature 0 $(\tilde{M}, \tilde{\nabla}^\circ, \tilde{g})$. Suppose M be \mathbf{R}^2 , and (∇, g) an induced statistical structure from $(\tilde{\nabla}, \tilde{g})$ by an immersion $f : (x, y) \in \mathbf{R}^2 \mapsto (0, x, y)$. We remark that (M, ∇, g) is a trivial statistical manifold of constant curvature 0.

Theorem 3.2 and Theorem 4.1 imply the following fact.

Corollary 4.1. *Let (M, ∇, g) be a trivial statistical manifold of constant curvature k , $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ a Hessian manifold of constant Hessian curvature \tilde{c} , and $f : M \rightarrow \tilde{M}$ a statistical immersion of codimension one. Then $f : M \rightarrow \tilde{M}$ is equiaffine, that is, τ^* vanishes.*

We consider a shape operator of statistical immersion of a trivial statistical manifold of constant curvature into a Hessian manifold of constant Hessian curvature.

Lemma 4.1. *Let (M, ∇, g) be a trivial statistical manifold of constant curvature k , $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ a Hessian manifold of constant Hessian curvature \tilde{c} , and $f : M \rightarrow \tilde{M}$ a statistical immersion of codimension one. Then the following holds:*

$$A^* = k\nu\tilde{c}^{-1}I, B^* = -\frac{1}{2}\nu I, h = \tilde{c}\nu^{-1}g, A = \tilde{c}\nu^{-1}I, B = [2\tilde{c}^2 - (2k + \tilde{c})\nu^2](2\nu\tilde{c})^{-1}I.$$

Proof. Combining Eq.(2.3) and Eq.(3.6) in [3] with Eq.(2.1), we have

$$\begin{aligned} \frac{\tilde{c}}{2}\{g(Y, Z)X - g(X, Z)Y\} &= 2(k - \tilde{k})\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad - b(Y, Z)A^*X + b(X, Z)A^*Y + h(X, Z)B^*Y - h(Y, Z)B^*X \\ 0 &= h(X, K(Y, Z)) - h(Y, K(X, Z)) + (\nabla_X b)(Y, Z) - (\nabla_Y b)(X, Z) \\ &\quad + \tau^*(X)b(Y, Z) - \tau^*(Y)b(X, Z) - \tau^*(Y)h(X, Z) + \tau^*(X)h(Y, Z) \\ 0 &= K(Y, A^*X) - K(X, A^*Y) - \tau^*(Y)A^*X + \tau^*(X)A^*Y \\ &\quad - (\nabla_X B^*)Y + (\nabla_Y B^*)X + \tau^*(X)B^*Y - \tau^*(Y)B^*X \\ 0 &= -h(X, B^*Y) + h(Y, B^*X) + (\nabla_X \tau^*)(Y) - (\nabla_Y \tau^*)(X) + b(Y, A^*X) - b(X, A^*Y) \end{aligned} \quad (4.6)$$

Taking the trace of (4.6)₁ with respect to X , we have

$$-\tilde{c}g(Y, Z) = -\text{tr}A^*b(Y, Z) + h(B^*Z, Y) + h(B^*Y, Z)$$

and taking the trace of (4.6)₁ with respect to Y , we have

$$-\frac{\tilde{c}}{2}(n+1)g(X, Z) = -b(A^*X, Z) + h(X, B^*Z) + \text{tr}B^*h(X, Z).$$

Using the above equation and Eq.(4.6)₄, we have

$$\begin{aligned} -\frac{\tilde{c}}{2}(n+2)g(X, Y) &= -b(A^*X, Y) + h(X, B^*Y) + \text{tr}B^*h(Y, Y) \\ &\quad - h(X, Y)\nu - h(X, B^*Y) + (\nabla_X \tau^*)Y + b(Y, A^*X) \\ &= \text{tr}B^*h(X, Y) - h(X, Y)\nu + (\nabla_X \tau^*)Y \end{aligned}$$

and since from Corollary 4.1 $\tau^* = 0$ holds, we have

$$(\nu - \text{tr}B^*)h(X, Y) = \frac{\tilde{c}}{2}(n+2)g(X, Y).$$

Hence we have

$$h = \frac{\tilde{c}}{2}(n+2)(\nu - \text{tr}B^*)^{-1}g. \quad (4.7)$$

If $\tilde{c} \neq 0$ holds, h is non-degenerated.

Since $\tilde{\nabla}$ is flat in Gaussian equation in [3], we obtain

$$k\{g(Y, Z)X - g(X, Z)Y\} = h(Y, Z)A^*X - h(X, Z)A^*Y$$

and taking the trace of above equation with respect to X , we have

$$k(n-1)g(Y, Z) = \text{tr}A^*h(Y, Z) - h(A^*Y, Z) = h((\text{tr}A^*I - A^*)Y, Z).$$

Since the above equation and Eq.(4.7) imply that

$$k(n-1)I = \frac{\tilde{c}}{2}(n+2)(\nu - \text{tr}B^*)^{-1}(\text{tr}A^*I - A^*),$$

there is $a \in \mathbf{R}$ such that $A^* = aI$ and $\text{tr}A^* = an$. Therefore the above equation implies that

$$k(n-1)I = \frac{\tilde{c}}{2}(n+2)(\nu - \text{tr}B^*)^{-1}(na - a)I$$

and thus since

$$2k(\nu - \text{tr}B^*) = \tilde{c}(n+2)a,$$

we have

$$A^* = 2k(\nu - \text{tr}B^*)[\tilde{c}(n+2)]^{-1}I. \quad (4.8)$$

If $k \neq 0$ holds, then since A^* is non-degenerated, by Eq.(4.8) we have

$$B^* = -\frac{\nu}{2}I, \text{tr}B^* = -\frac{n\nu}{2}$$

and

$$A^* = \frac{2k(\nu + \frac{n\nu}{2})}{\tilde{c}(n+2)}I = \frac{k\nu}{\tilde{c}}I, h = \frac{\tilde{c}}{2}(n+2)(\nu + \frac{n\nu}{2})^{-1}g = \frac{\tilde{c}}{\nu}g.$$

Since $h(X, Y) = g(AX, Y)$, we have $A = \frac{\tilde{c}}{\nu}I$ and

$$B = B^* + (A - A^*) = -\frac{\nu}{2}I + (\frac{\tilde{c}}{\nu} - \frac{k\nu}{\tilde{c}})I = \frac{-\nu^2\tilde{c} + 2\tilde{c}^2 - 2k\nu^2}{2\nu\tilde{c}}I = \frac{2\tilde{c}^2 - (2k + \tilde{c})\nu^2}{2\nu\tilde{c}}I.$$

□

Theorem 4.2. *Let (M, ∇, g) be a trivial statistical manifold of constant curvature k , $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ a Hessian manifold of constant Hessian curvature \tilde{c} . If there is a statistical immersion of codimension one $f : M \rightarrow \tilde{M}$, $2k + \tilde{c}$ is of non-negative. Moreover, when \tilde{c} is positive, the Riemannian shape operator of $f : M \rightarrow \tilde{M}$ is given by $S = \pm \frac{1}{2}\sqrt{2k + \tilde{c}}I$.*

Proof. By Lemma 4.1 and Eq.(4.2), we have

$$\begin{aligned} & \frac{\tilde{c}}{4}\{g(Y, Z)X - g(X, Z)Y\} + \frac{2\tilde{c}^2 - (2k + \tilde{c})\nu^2}{2\nu\tilde{c}}(-\frac{\nu}{2})\{g(Y, Z)X - g(X, Z)Y\} \\ &= \left[\frac{\tilde{c}}{4} - \frac{2\tilde{c}^2 - (2k + \tilde{c})\nu^2}{4\tilde{c}} \right] \{g(Y, Z)X - g(X, Z)Y\} = 0 \end{aligned}$$

and thus conclude that

$$\frac{\tilde{c}}{4} - \frac{2\tilde{c}^2 - (2k + \tilde{c})\nu^2}{4\tilde{c}} = 0.$$

Since $\tilde{c}^2 = (2k + \tilde{c})\nu^2$, we have $2k + \tilde{c} \geq 0$ and

$$\nu = \pm \frac{|\tilde{c}|}{\sqrt{2k + \tilde{c}}}.$$

Thus the Riemannian shape operator S is given by

$$S = A^* - B^* = (\frac{k\nu}{\tilde{c}} + \frac{\nu}{2})I = \frac{2k + \tilde{c}}{2\tilde{c}}(\pm \frac{|\tilde{c}|}{\sqrt{2k + \tilde{c}}})I = \pm \frac{|\tilde{c}|}{2\tilde{c}}\sqrt{2k + \tilde{c}}I.$$

When \tilde{c} is positive, we have $S = \pm \frac{1}{2}\sqrt{2k + \tilde{c}}I$.

□

Example 4.2. Let $(H, \tilde{\nabla}, \tilde{g})$ be the $(n+1)$ -dimensional upper half Hessian space of constant Hessian curvature 4 as in Example 2.1. For a constant $y_0 > 0$, write the following immersion by f :

$$(y^1, \dots, y^n)^T (\in \mathbf{R}^n) \mapsto (y^1, \dots, y^n, y_0)^T \in H.$$

Let (∇, g) be the statistical structure on \mathbf{R}^n induced by f from $(\tilde{\nabla}, \tilde{g})$. Then $(\mathbf{R}^n, \nabla, g)$ is a trivial statistical manifold of constant curvature 0 and f is a statistical immersion of a trivial statistical manifold of constant curvature into Hessian manifold of constant Hessian curvature.

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